

# Critical exponents of the 3d Ising and related models from Conformal Bootstrap

---

**Ferdinando Gliozzi<sup>a,b</sup>, Antonio Rago<sup>a</sup>**

<sup>a</sup> *School of Computing and Mathematics and Centre for Mathematical Science,  
Plymouth University, Plymouth PL4 8AA, United Kingdom*

<sup>b</sup> *Dipartimento di Fisica, Università di Torino  
and Istituto Nazionale di Fisica Nucleare - sezione di Torino  
Via P. Giuria 1, I-10125 Torino, Italy*

**ABSTRACT:** The constraints of conformal bootstrap are applied to investigate a set of conformal field theories in various dimensions. The prescriptions can be applied to both unitary and non unitary theories allowing for the study of the spectrum of low-lying primary operators of the theory. We evaluate the lowest scaling dimensions of the local operators associated with the Yang-Lee edge singularity for  $2 \leq D \leq 6$ . Likewise we obtain the scaling dimensions of six scalars and four spinning operators for the 3d critical Ising model. Our findings are in agreement with existing results to a per mill precision and estimate several new exponents.

**KEYWORDS:** Conformal Field Theory, Conformal Bootstrap, Ising Model.

---

## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Conformal Bootstrap Constraints</b>	<b>4</b>
<b>3. The Yang-Lee model</b>	<b>8</b>
<b>4. 3d critical Ising model</b>	<b>11</b>

---

## 1. Introduction

Understanding conformal field theories (CFTs) is a great challenge of many branches in theoretical physics. CFTs are at the heart of critical phenomena in condensed matter physics, they control the renormalisation group flows of quantum field theories and explain the appearance of universal scaling laws [1, 2]; they even provide a tool to study quantum gravity via the AdS/CFT correspondence [3].

One of the main properties characterizing a specific CFT is the spectrum of the scaling dimensions of its local operators. With the exception of some two-dimensional CFTs (where the algebra of conformal generators is infinite-dimensional) calculating these quantities is very challenging, as they are dominated by quantum fluctuations, an effect which takes place on all length scales in these scale-invariant theories. Moreover, most CFTs are strongly coupled and difficult to study using the usual perturbative techniques of Feynman diagrams, although some of them can be accurately analysed by Monte Carlo calculations and/or strong coupling expansions [7, 8, 9].

Conformal bootstrap is a reincarnation of the bootstrap mechanism used to investigate the CFT constraints originating from the crossing symmetry of the four-point functions. It was suggested a long time ago that conformal bootstrap could give useful informations on the allowed scaling dimensions of the theory [4, 5, 6]. This idea was implemented in two-dimensional rational CFTs, i.e. those with a finite number of Virasoro primary fields. It was shown that the crossing symmetry, when combined with the modular invariance of the theory on a torus, provides the complete spectrum of scaling dimensions, modulo an integer [10].

In recent years it has been shown that the conformal bootstrap approach can give accurate predictions for specific CFTs in any space-time dimension [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The starting point of this active field of research was the observation [11] that the conformal bootstrap equations (i.e the functional equations following from the crossing symmetry) can be rewritten as an infinite system of linear homogeneous equations. In order to study the system numerically the number of unknowns

and the number of equations has to be truncated. In the simplest case, i.e. the four-point function of a single scalar field  $\phi$  with scaling dimension  $\Delta_\phi$ , the truncated bootstrap equations take the form

$$\sum_{i=1}^N f_e^i \mathbf{p}_i = 0, \quad (e = 1, 2, \dots, M), \quad (1.1)$$

where the unknown  $\mathbf{p}_i$  is the square of the coefficient of the primary operator  $\mathcal{O}_i$  contributing to the operator product expansion (OPE) of  $\phi(x)\phi(y)$ . The coefficient  $f_e^i$  – made up of multiple derivatives of generalized hypergeometric functions – depends on  $\Delta_\phi$  and on the scaling dimension  $\Delta_i$  of  $\mathcal{O}_i$ . Unitarity requires  $\mathbf{p}_i \geq 0$ ; this implies that the search of solutions for eq. (1.1), when combined with a normalization condition, can be reformulated as a linear programming problem, which can be treated numerically. Its solution yields a numerical upper bound  $\Delta_u = f(\Delta_\phi)$  on the dimension of the first scalar contributing to the OPE of  $\phi(x)\phi(y)$  [11]. Some CFTs may exist that saturate this bound [15, 21]. Since the output of the linear programming algorithm is a solution of eq. (1.1), the low-lying spectrum of operator dimensions for these special CFTs can be evaluated. The accuracy of these calculations is particularly impressive in the case of the two-dimensional critical Ising model, where a comparison with exact results can be made. A drawback of this approach is that it can be applied only to theories saturating the unitarity bound. Moreover there is no physically convincing ground explaining why some CFTs saturate the bound<sup>1</sup>.

Recently one of us has proposed a different approach to solve numerically the bootstrap constraints (1.1) which can be applied to a larger class of CFTs [20]. The starting point was the observation that any finite truncation of the bootstrap equations results in a set constraints on the spectrum of operator dimensions contributing to eq. (1.1), provided that the number  $M$  of equations is equal or larger than the number  $N$  of unknowns. In this case the homogeneous system admits a non-identically vanishing solution if and only if all the minors of order  $N$  are vanishing. Hence for any subset of  $N$  equations the constraint can be expressed as

$$\det \mathbf{f}_n \equiv f_n(\Delta_\phi, \Delta_1, \Delta_2, \dots, \Delta_N) = 0, \quad (n = 1, 2, \dots, \binom{M}{N}) \quad (1.2)$$

where  $\mathbf{f}_n$  is the  $N \times N$  matrix of coefficients of the  $n$ th subset. As soon as  $M$  exceeds  $N$ , the number of constraints is equal or larger than  $N$ ; the all set of constraints can be used to extract the scaling dimensions of the primary operators involved, if a solution exists. If the system of constraints is over-determined, i.e. there are more independent equations than unknowns, it can be split in consistent subsystems to obtain a set of solutions; their spread provides a rough estimate of the error.

This method is quite general and it can be applied to any CFT, whether unitary or not, provided the theory has a discrete spectrum and it admits a consistent truncation (1.1) of the bootstrap equations. This statement hides a subtlety that deserves to be mentioned.

---

<sup>1</sup>In section 2 the condition for an actual CFT to saturate this unitarity bound is rewritten in a closed form.

The number  $M$  of homogeneous equations is not an independent parameter: indeed the larger is the number  $N$  of terms in (1.1), the better is the approximation of the rhs to a constant, and hence the larger is the number  $M$  of approximately vanishing derivatives.

The method outlined above is valid only if it exists an  $N_0$  such that  $M > N_0$ . If this happens for small values of  $N_0$  we say that the CFT is easily truncable. Typical examples are the free scalar massless theories in  $D$  dimensions, where the method yields very accurate results already for  $N_0 = 3$  [20]. In spite of the fact that the conformal block expansion shows good convergence properties [16], there are cases in which the assumption  $M > N$  can not be consistently made for any value of  $N$  (an explicit example is shown in the next section). In these cases the associated CFT is not truncable and the numerical methods [11] or [20] do not apply.

Of course the method is predictive only if the exact form of the four-point function for a given CFT is not known. Should this be the case, we do not know *a priori* whether the theory is easily truncable. The symmetry properties of the model can be used to guess the structure of the quantum numbers of the low-lying primary operators. If a valid set of zeros for  $N$  determinants of the type (1.2) is found, we can say *a posteriori* that the CFT is easily truncable. The larger is  $N$  the smaller is the error of the estimate of the operator dimensions. Thus, once a solution is found, it can be improved by adding new operators in the game. In this way the low-lying spectrum of the Yang-Lee model in several dimensions as been evaluated in [20] and will be further improved in this paper.

A major result presented in this paper is an exact solution of the conformal bootstrap constraints (1.2) with  $N = 7$  primary operators and  $M = 8$  equations. It represents a consistent truncation of the four-point function  $\langle \sigma(x_1)\sigma(x_2)\sigma(x_3)\sigma(x_4) \rangle$  of the 3d Ising model at the critical point described by the spontaneous breakdown of the  $\mathbb{Z}_2$  symmetry.  $\sigma(x)$  is the  $\mathbb{Z}_2$ -odd scalar, i.e. the order parameter of the theory. The set of primary operators comprises: four  $\mathbb{Z}_2$ -even scalars, a conserved spin 2 operator, a quasi-conserved spin 4 operator and a spin 6 operator. One of the four  $\mathbb{Z}_2$ -even scalars is identified with the energy  $\varepsilon$ , the only relevant operator beside  $\sigma$ , while the other three are recurrences,  $\varepsilon', \varepsilon'', \varepsilon'''$ , whose dimensions are related to the critical exponents measuring the corrections to scaling. The spin 2 operator is identified with the stress tensor, while the spin 4 operator is related to the critical exponent associated with the rotational symmetry breaking. It turns out that the solution depends on a free parameter. Using the scaling dimension of the spin 4 operator as an input all the other operator dimensions can be evaluated and they differ from the best estimates for no more than 1 %. To the best of our knowledge the scaling dimension of  $\varepsilon'''$  has never estimated before.

Assuming this truncation is not only a solution of crossing symmetry constraints, but a part of a full-fledged CFT, we exploit some consistency checks to further enlarge the number of estimated scaling dimensions of primary operators belonging to the spectrum of the 3d critical Ising model, as reported in Tables 4 and 5.

Even though a formal proof of conformality of the critical point of the 3d Ising model is still missing [24], numerical simulations and theoretical arguments leave little doubt on its validity. A new, strong support of this conjecture comes from the striking agreement between know data and our results, based on the assumption of conformal invariance of

the 3d Ising critical point .

The plan of the paper is the following. In the next Section we fix the notations and discuss the derivation of the homogeneous system (1.1) in relation with the notion of easily truncable CFT. Section 3 is dedicated to Yang-Lee models in various dimensions while the last Section is devoted to the 3d critical Ising model and to some conclusions.

## 2. Conformal Bootstrap Constraints

The four-point function of a scalar  $\phi(x)$  of a  $D$ -dimensional CFT can be parametrised as [1]

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{g(u, v)}{|x_{12}|^{2\Delta_\phi} |x_{34}|^{2\Delta_\phi}}, \quad (2.1)$$

where  $\Delta_\phi$  is the scaling dimension of  $\phi$ ,  $x_{ij}^2$  is the square of the distance between  $x_i$  and  $x_j$ ,  $g(u, v)$  is a function of the cross-ratios  $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$  and  $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ .  $g(u, v)$  can be expanded in terms of the conformal blocks  $G_{\Delta, L}(u, v)$ , i.e. the eigenfunctions of the Casimir operator of  $SO(D+1, 1)$ :

$$g(u, v) = 1 + \sum_{\Delta, L} \mathbf{p}_{\Delta, L} G_{\Delta, L}(u, v), \quad (2.2)$$

with  $\mathbf{p}_{\Delta, L} = \lambda_{\phi\phi\mathcal{O}}^2$ , where  $\lambda_{\phi\phi\mathcal{O}}$  is the coefficient of the primary operator  $\mathcal{O}$  of scaling dimension  $\Delta$  and spin  $L$  contributing to the OPE of  $\phi(x)\phi(y)$ .

The lhs of (2.1) is invariant under any permutation of the  $x_i$ 's while the rhs is not, unless  $g(u, v)$  obeys the following two functional equations

$$g(u, v) = g(u/v, 1/v); \quad v^{\Delta_\phi} g(u, v) = u^{\Delta_\phi} g(v, u). \quad (2.3)$$

The first equation projects out the odd spins in the expansion (2.2). The second one, after separating the identity from the other primary operators, can be rewritten as a sum rule

$$\sum_{\Delta, L} \mathbf{p}_{\Delta, L} \frac{v^{\Delta_\phi} G_{\Delta, L}(u, v) - u^{\Delta_\phi} G_{\Delta, L}(v, u)}{u^{\Delta_\phi} - v^{\Delta_\phi}} = 1. \quad (2.4)$$

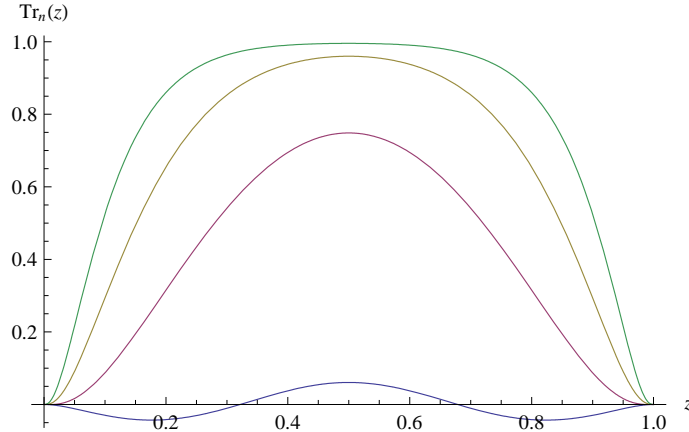
It is easy to argue that such a relation can be exactly satisfied only if the sum contains infinite terms [11]. In any finite truncation the rhs of (2.4) is only approximately constant. In order to transform the sum rule into the system (1.1) of linear homogeneous equations one has to assume that the derivatives of the rhs are vanishing, an assumption that in some cases cannot be made. As an example, consider the four-point function of the energy operator :  $\phi^2(x) := \lim_{\epsilon \rightarrow 0} \phi(x)^i \phi_i(x + \epsilon) - \langle \phi^j(x) \phi_j(x + \epsilon) \rangle$  in a four-dimensional  $O(n)$ -invariant free field theory ( $i, j = 1, 2, \dots, n$ ).

We have

$$g_n(u, v) = 1 + \frac{4}{n} \left( u + \frac{u}{v} + \frac{u^2}{v} \right) + \frac{u^2}{v^2} + u^2. \quad (2.5)$$

The first few terms of the conformal block expansion are

$$g_n(u, v) = 1 + \frac{8}{n} G_{2,0} + \left(2 + \frac{4}{n}\right) G_{4,0} + \frac{16}{3n} G_{4,2} + \left(2 - \frac{2}{n}\right) G_{6,0} + \frac{24 + \frac{8}{n}}{5} G_{6,2} + \frac{64}{35n} G_{6,4} + \dots \quad (2.6)$$



**Figure 1:** Progressive truncations of the sum rule (2.4) in the case of 4-point function of the operator energy in four-dimensional  $O(n)$ -invariant free massless scalar theory. The four curves represent respectively the contribution of primaries of scaling dimensions  $\Delta \leq 4$ ,  $\Delta \leq 6$ ,  $\Delta \leq 8$ ,  $\Delta \leq 10$ . In this specific example we have chosen  $n = 1$ .

Figure 1 shows the first few truncations. We used the parametrisation  $u = z\bar{z}$  and  $v = (1 - z)(1 - \bar{z})$  [25]. Clearly these truncations are not flat enough around the symmetric point  $z = \bar{z} = \frac{1}{2}$ . In this case it would be erroneous to guess that higher level truncations could always generate a number  $M$  of homogeneous equations sufficient to solve the system. Actually this four-point function is not truncable. The proof is suprisingly simple: if  $M$  is larger than  $N$ , then the knowledge of the operator spectrum should determine the  $\mathbf{p}_{\Delta,L}$ 's, but these depend on  $n$  (see eq. (2.6)), while the spectrum of primaries of such a free field theory does not depend on it.

The easily truncable CFTs are instead those where the first few terms suffice to give an almost constant rhs in (2.4). This is the case for instance of free scalar massless theories in  $D$  dimensions. Figure 2 shows the first few truncations at  $D = 3$ , where

$$g(u, v) - 1 \equiv \sqrt{u} + \sqrt{\frac{u}{v}} = 2G_{1,0} + \frac{1}{4}G_{3,2} + \frac{1}{64}G_{5,4} + \frac{1}{1024}G_{7,6} + \dots \quad (2.7)$$

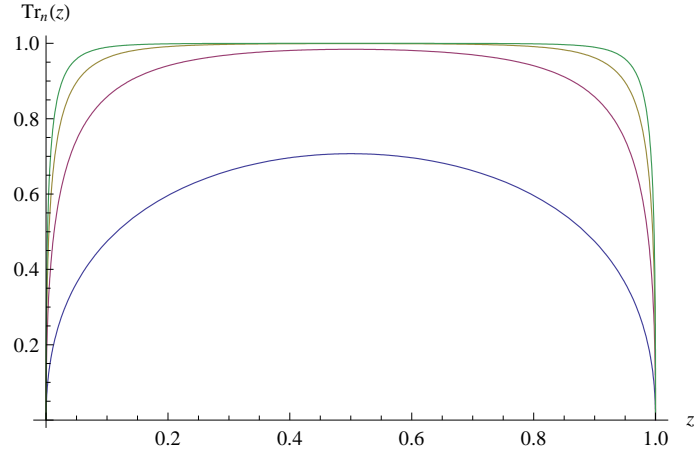
In these cases one can follow the procedure of [11], i.e. Taylor expand (2.4) about the symmetric point  $z = \bar{z} = \frac{1}{2}$  and transform a  $N$ -terms truncation of the sum rule into a set of  $M$  linear equations in  $N$  unknowns  $\mathbf{p}_{\Delta,L}$ .

To be more specific, following [15] we make the change of variables  $z = (a + \sqrt{b})/2$ ,  $\bar{z} = (a - \sqrt{b})/2$  and Taylor expand around  $a = 1$  and  $b = 0$ . It is easy to see that this expansion will contain only even powers of  $(a - 1)$  and integer powers of  $b$ . The truncated sum rule can then be rewritten as one inhomogeneous equation

$$\sum_{\Delta,L} \mathbf{p}_{\Delta,L} f_{\Delta,\Delta_L}^{(0,0)} = 1, \quad (2.8)$$

and a set of  $M$  homogeneous equations already mentioned in eq. (1.1) and here rewritten in more details

$$\sum_{\Delta,L} f_{\Delta,\Delta_L}^{(2m,n)} \mathbf{p}_{\Delta,L} = 0, \quad (m, n \in \mathbb{N}, m + n \neq 0), \quad (2.9)$$



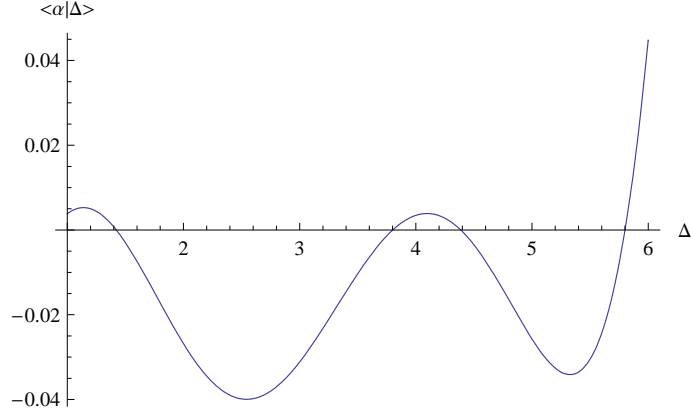
**Figure 2:** Progressive truncations of the sum rule (2.4) in the case of 4-point function of a free massless scalar theory in three dimensions. Few terms are sufficient to well approximate the rhs of the sum rule around the symmetric point  $z = \frac{1}{2}$ .

with

$$\mathbf{f}_{\alpha,\beta}^{(m,n)} = \left( \partial_a^m \partial_b^n \frac{v^\alpha G_\beta(u,v) - u^\alpha G_\beta(v,u)}{u^\alpha - v^\alpha} \right)_{a=1,b=0}. \quad (2.10)$$

The number  $M$  of homogeneous equations depends on the degree of flatness of the truncation. When  $M > N$  the system (2.9) becomes highly predictive, since it admits a non identically vanishing solution if and only if all the minors of order  $N$  are vanishing. The common intersection of these zeros identify the spectrum values of the scaling dimensions for the  $N$  primary operators. If there are more independent minors than unknowns we get a set of scattered solutions. Their spread gives a rough estimate of the error of the approximation. Inserting these  $\Delta$ 's in eq. (2.9) and assuming that the corresponding minor has rank  $N - 1$  (i.e. a simple zero) we obtain a one-parameter family of  $\mathbf{p}_{\Delta,L}$ 's. The inhomogeneous equation (2.8) sets their normalization.

It is instructive to consider at the homogeneous system (2.9) from a slightly different angle. The  $\mathbf{p}_{\Delta,L}$ 's can be viewed as the components of the right-eigenvector with zero eigenvalue of  $\mathbf{f}$ . In Dirac notation we can write  $\mathbf{f}|\mathbf{p}\rangle = 0$ . Of course there is also a left-eigenvector associated with the same eigenvalue, i.e.  $\langle\alpha|\mathbf{f} = 0$ . It can be noticed that  $\langle\alpha|$  encodes the complete information on the low-lying spectrum of scaling dimensions of the theory. Indeed  $\langle\alpha|$  is by construction orthogonal to the  $N$  columns of  $\mathbf{f}$ . Each column depends only on a single conformal block, so it can be viewed as a vector that we denote as  $|L, \Delta, \Delta_\phi\rangle$ . By the knowledge of  $\langle\alpha|$  it is possible to reconstruct the whole spectrum of the  $N$  primary operators as follows. Since the stress-tensor is always present among these operators, the zeros of  $f(x) \equiv \langle\alpha|2, D, x\rangle$  give the possible values of  $\Delta_\phi$ . It follows that the zeros of the scalar product  $\langle\alpha|L, \Delta, \Delta_\phi\rangle$ , as a function of  $\Delta$  and for fixed  $L$  and  $\Delta_\phi$ , provide the whole spectrum (see an example in figure 3). In conclusion, the combination of the two eigenvectors  $|\mathbf{p}\rangle$  and  $\langle\alpha|$  encodes the complete information to reconstruct the (truncated) four-point function for the CFT under study.



**Figure 3:** The scalar product  $\langle \alpha | \Delta \rangle \equiv \langle \alpha | 0, \Delta, \Delta_\phi \rangle$  as a function of  $\Delta$  in the critical 3d Ising model.  $\langle \alpha |$  is the left eigenvector corresponding to the null eigenvalue of the  $7 \times 7$  matrix  $\mathbf{f}$  discussed in Section 4 and  $|\Delta\rangle$  is a column of  $\mathbf{f}$  associated with conformal blocks of spin 0. The zeros are the scaling dimensions of the  $\mathbb{Z}_2$ -even scalars  $\varepsilon, \varepsilon', \varepsilon'', \varepsilon'''$ .

The eigenvector  $\langle \alpha |$  bears some similarity to the linear functional  $\langle \Lambda |$  discussed in [11] in relation with the unitarity upper bound. In both cases the zeros of the scalar product with  $|L, \Delta, \Delta_\phi\rangle$  provide the operator spectrum [14, 18], hence a solution to the crossing symmetry; on the other hand  $\langle \Lambda |$  must fulfil the much stronger constraint

$$\langle \Lambda | L, \Delta, \Delta_\phi \rangle \geq 0, \quad \Delta \geq \Delta_L^o. \quad (2.11)$$

$\Delta_L^o$  is the upper unitarity bound described in the introduction: if at a given spin  $L$  the first primary operator  $\mathcal{O}_{\Delta,L}$  has scaling dimension larger than this bound, the theory is not unitary. Thus if a unitary CFT saturates the bound, the scalar product above must vanish at  $\Delta = \Delta_L^o$  and all the recurrences at the same  $L$  must correspond to double zeros of (2.11), i.e.

$$\langle \Lambda | L, \Delta, \Delta_\phi \rangle = 0, \quad \langle \Lambda | \frac{d}{d\Delta} | L, \Delta, \Delta_\phi \rangle = 0, \quad \Delta \in \Sigma, \Delta > \Delta_L^o, \quad (2.12)$$

where  $\Sigma$  denotes the spectrum of allowed operator dimensions.

It follows that  $\langle \Lambda |$  can be seen as a left eigenvector of zero eigenvalue for a matrix  $\mathbf{F}$  much larger than  $\mathbf{f}$ , composed by the columns  $|L, \Delta, \Delta_\phi\rangle$  for all  $\Delta \in \Sigma$  and the columns  $\frac{d}{d\Delta} |L, \Delta, \Delta_\phi\rangle$  for all recurrences. For instance, if the set of  $N$  operators of a given truncation of a CFT is composed by  $n_0$  scalars,  $n_2$  operators of spin 2 and so on, so that  $N = n_0 + n_2 + \dots + n_{2k}$ , the matrix  $\mathbf{F}$  has  $2n_0 - 1$  columns for the scalars,  $2n_2 - 1$  columns for the spin 2, and so on, so that the total number of columns is  $N_F = 2N - k$ .

Providing a solution to the crossing symmetry is equivalent to finding a  $N$ -dimensional left eigenvector  $\langle \alpha | \mathbf{f} = 0$ . The additional constraint of saturating the unitarity bound requires instead to find a solution of  $\langle \Lambda | \mathbf{F} = 0$ . If  $\langle \Lambda |$  is unique as claimed in [18], then  $\mathbf{F}$  is a square  $N_F \times N_F$  matrix with  $\det \mathbf{F} = 0$  and rank  $N_F - 1$ .

In this paper we only study the zeros of  $\det \mathbf{f}$ . An essential ingredient to pursuing high accuracy in these calculations is an efficient method to evaluate with high precision



the conformal blocks and their derivatives. These are known in a closed form only for  $D = 2, 4, 6$  [25]. The conformal blocks for generic  $D$  can be still rewritten in a closed form for  $z = \bar{z}$ ,  $L = 0$  and  $L = 1$  in terms of  ${}_3F_2$  hypergeometric functions

$$G_{\Delta,0}(z) = \left(\frac{z^2}{1-z}\right)^{\Delta/2} {}_3F_2 \left[ \frac{\Delta}{2}, \frac{\Delta}{2}, \frac{\Delta}{2} - \alpha; \frac{\Delta+1}{2}, \Delta - \alpha; \frac{z^2}{4(z-1)} \right], \quad (2.13)$$

$$G_{\Delta,1}(z) = \left(\frac{z^2}{1-z}\right)^{\frac{\Delta+1}{2}} {}_3F_2 \left[ \frac{\Delta+1}{2}, \frac{\Delta+1}{2}, \frac{\Delta+1}{2} - \alpha; \frac{\Delta}{2} + 1, \Delta - \alpha; \frac{z^2}{4(z-1)} \right], \quad (2.14)$$

with  $\alpha = \frac{D}{2} - 1$ .

We follow the algorithms developed in [15] which allow to write, through the use of a few recursion relations, each matrix coefficient of (2.9) as

$$\mathbf{f}_{\Delta_\phi, \Delta_L}^{(2m,n)} = \sum_{i=1}^6 R_i(D, L, \Delta_L, \Delta_\phi) \mathcal{B}_i(D, \Delta_L, z = \frac{1}{2}), \quad (2.15)$$

where the six basis functions  $\mathcal{B}_i$  are the two conformal blocks  $G_{\Delta,0}(z)$  and  $G_{\Delta,1}(z)$  and their first and second derivatives with respect to  $z$ ; the  $R_i$ 's are rational functions of their arguments.

These formulas work for any value of dimension, even fractional ones. As such they have already been used to study conformal bootstrap in non-integer dimensions [22]. Here we use this property when comparing our results with the epsilon expansion of the Yang-Lee model in  $6 - \epsilon$  dimensions.

### 3. The Yang-Lee model

Besides the CFT describing its critical point, there are two other CFTs, at least, related to the 3d Ising model. One is the one-dimensional CFT associated with a line defect of this model. The low-lying spectrum of scaling dimensions of the local operators living on the defect has been estimated using Monte Carlo simulations only recently [26]; these estimates have been supported by both epsilon expansion and conformal bootstrap calculations [23].

The other CFT is much older and originates from two seminal papers of Yang and Lee [27, 28], dated more than sixty years ago, on the analytic properties of phase transitions. It turns out that the zeros of the partition function of a ferromagnetic Ising model in  $D$  dimensions in a magnetic field  $h$  are located on the imaginary axis above a critical value  $h_c(T)$  called the Yang-Lee edge singularity. Above the critical temperature  $T > T_c$ , in the thermodynamic limit, the density of these zeros behaves near  $h_c$  like  $(h - h_c)^\sigma$ , where the edge exponent  $\sigma$  is universal and characterizes a universality class which is not related to the spontaneous breakdown of any symmetry. The Yang-Lee universality class can be described by the non-trivial fixed point of a  $\phi^3$  theory [29] with imaginary coupling, thus the corresponding CFT is non-unitary hence it cannot be studied with the conformal bootstrap approach of [11].

The critical exponent  $\sigma$  is related to the scaling dimension of  $\phi$  by

$$\sigma(D) = \frac{\Delta_\phi}{D - \Delta_\phi}. \quad (3.1)$$

The interest on the Yang-Lee universality class is enhanced by the discovery, in the past years, that the edge singularity is related to other quite different critical behaviours. For instance, the number per site of large isotropic branched polymers in a good solvent (i.e. *lattice animals*) obeys a power law with exponent  $\varphi_I(D) = \sigma(D-2) + 2$  [30], or the pressure of  $D$ -dimensional fluids with repulsive core has a singularity at negative values of activity with universal exponent  $\varphi(D) = \sigma(D) + 1$  [31].

Monte Carlo simulations and other numerical methods on these systems gave accurate results for  $\sigma$  [32, 33]. Recent high temperature expansions of the Ising model in low magnetic field have further improved the accuracy in the whole range  $2 < D < 6$  [34].

The edge exponent  $\sigma$  is exactly known in  $D = 2$  and  $D = 6$  dimensions. For the first case the exact form of the four-point function of  $\phi(x)$  has been found [35]; we shall use this result to test the accuracy of our method.  $D = 6$  is the upper critical dimensionality of the  $\phi^3$  model, above which the classical mean-field value  $\sigma = \frac{1}{2}$  applies.

Our approach to conformal bootstrap only requires to know the spin and possibly other quantum numbers of the low-lying primary operators contributing to the OPE of  $\phi(x)\phi(y)$ . It is not necessary to know the detailed form of this OPE but simply its fusion rule that we write as

$$[\Delta_\phi] \times [\Delta_\phi] = \sum_i [\Delta_i, L_i], \quad (3.2)$$

where  $[\Delta, L]$  denotes the primary operator with scaling dimension  $\Delta$  and spin  $L$  and we set  $[\Delta] = [\Delta, 0]$ . Note that this is a shorthand way of writing the conformal block expansion (2.2) or even the first terms of a Clebsch-Gordan series of the conformal group  $SO(D+1, 1)$ . In a free scalar theory in  $D$  dimensions we have (see for instance (2.7))

2d Yang Lee model – Truncated Fusion Rules					Exact results
	$[\Delta_\phi] \times [\Delta_\phi] = 1 + [\Delta_\phi] + [2, 2] + [\Delta_4, 4]$	$+ [\Delta_6, 6] + [6, 6]$	$+ [\Delta_6, 6] + [\Delta']$	$+ [6, 6] + [\Delta']$	
$\Delta_\phi$	-0.385(7)	-0.40033(1)	-0.40062(3)	-0.39777(2)	$-\frac{2}{5}$
$\Delta_4$	3.70(4)	3.5904(3)	3.58961(5)	3.5963(2)	$\frac{18}{5}$
$\Delta_6$	–	5.593(3)	5.590(3)	–	$\frac{28}{5}$
$\Delta'$	–	–	7.60(1)	7.61(8)	$\frac{38}{5}$
$c$	-4.5(1)	-4.38(1)	-4.38(1)	-4.44(1)	$-\frac{22}{5}$
$p_\phi$	-3.67(1)	-3.6524(3)	-3.6524(3)	-3.6557(3)	-3.65312..

**Table 1:** The low-lying spectrum of primary operators of the 2d Yang-Lee model estimated for 4 different truncations of the fusion rule (3.4) and compared with the exact results (last column). A rough estimate of the errors is obtained by the spread of the solutions generated by different set of equations. An example is depicted in figure 4.

$$[\Delta_\phi] \times [\Delta_\phi] = 1 + [\Delta_{\phi^2}] + [D, 2] + [D + 2, 4] + [D + 4, 6] + \dots, \quad (3.3)$$

where  $[D + L - 2, L]$  is a conserved spinning operator. In an interacting theory only the spin 2 – the stress-tensor  $T_{\mu\nu}$  – is generally conserved. Moreover in the  $\phi^3$  theory  $\phi^2$  is a redundant operator, since at the non-trivial fixed point in  $D < 6$  dimensions it

is proportional to  $\partial^2\phi$  by the equation of motion. Thus  $\phi^2$  and its derivatives become descendant operators of the only relevant operator  $\phi$  of the Yang-Lee universality class. Thus the fusion rule that characterizes the low-lying spectrum of this CFT is expected to be

$$[\Delta_\phi] \times [\Delta_\phi] = 1 + [\Delta_\phi] + [D, 2] + [\Delta_4, 4] + [\Delta'] + \dots, \quad (3.4)$$

where  $[\Delta']$  in the perturbative renormalisation group analysis corresponds to  $\phi^3$ . This scalar cannot be neglected if we include in the fusion rule the spin 4 operator, since in  $6 - \epsilon$  dimensions  $\Delta_{\phi^3} < \Delta_4$ . Of course we expect also a term  $[\Delta_6, 6]$ , like in a free-field theory, however its contribution becomes important only in the two-dimensional case.

Comparison with the exact solution at  $D = 2$  shows that there are actually two primary operators of spin 6. One of them is conserved and coincides with  $T^3$ . Inserting truncations of such a fusion rule in eq. (1.2) we found only isolated solutions where the number  $M$  of fulfilled equations is larger than the number  $N$  of unknowns  $\Delta$ 's, thus this CFT is easily truncable.

In table 1 we report the results of a truncation of this fusion rule to the first 4 terms and three different truncations of 6 terms. As expected the accuracy increases with the number of included conformal blocks. Inserting these  $\Delta$ 's in the linear system (2.8) and (2.9) we can evaluate the OPE coefficients, in particular  $\mathbf{p}_{\Delta_\phi}$  that turns out to be very close to the known exact result. Similarly from  $\mathbf{p}_{2,2}$  we can extract the central charge  $c = \Delta_\phi^2 / \mathbf{p}_{2,2}$ .

We encountered isolated solutions also in  $D > 2$ . The estimates for  $D \leq 6$  generated by the truncated fusion rule (3.4) are reported in tables 2 and 3.

$D$ dimensional Yang-Lee model—the edge exponent $\sigma$					
$D$	bootstrap	Ising in $H$	Fluids	Animals	$\epsilon$ —expansion
2	-0.1664(5)	-0.1645(20)	-0.161(8)	-0.165(6)	(exact -1/6)
3	0.085(1)	0.077(2)	0.0877(25)	0.080(7)	0.079-0.091
4	0.2685(1)	0.258(5)	0.2648(15)	0.261(12)	0.262-0.266
5	0.4105(5)	0.401(9)	0.402(5)	0.40(2)	0.399-0.400
6	0.4999(1)	0.460(50)	0.465(35)	—	1/2

**Table 2:** Estimates of  $\sigma(D)$  from the truncated fusion rule (3.4) compared with the best estimates from strong coupling expansions [34], fluids [32] and lattice animals [33]. A more complete list of estimates can be found in [34]. The bootstrap result for  $D = 2$  is calculated from tab 1.

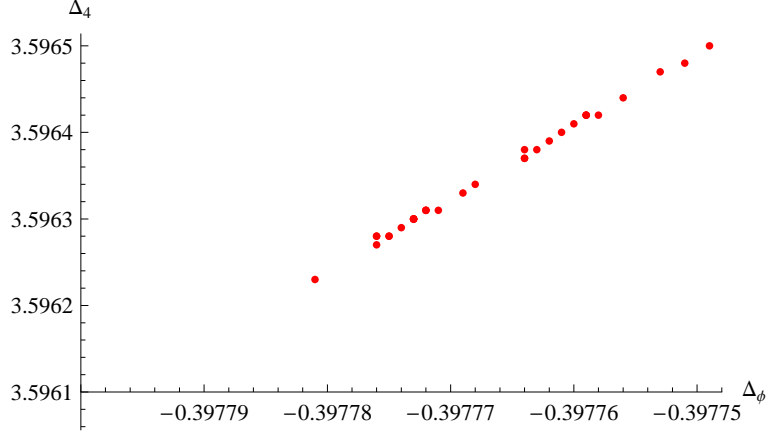
As we have already noticed, the present method works for any value of dimension. As such it is interesting to follow the flow of the spectrum from the six-dimensional free-field theory in  $6 - \epsilon$  dimensions. The epsilon expansion of  $\sigma(D)$  known up to, and including,  $\epsilon^3$  contributions is [36]

$$\sigma(6 - \epsilon) = \frac{1}{2} - \frac{1}{12}\epsilon - \frac{79}{3888}\epsilon^2 + \left( \frac{\zeta(3)}{81} - \frac{10445}{1250712} \right) \epsilon^3 + O(\epsilon^4). \quad (3.5)$$

In figure 5 we plot this function in the range  $5 \leq D \leq 6$  as well as the bootstrap estimates.

$D$ dimensional YL model– $\Delta_4$ and $\Delta'$			
$D$	$\mathbf{p}_\phi$	$\Delta_4$	$\Delta'$
3	-3.88(1)	4.75(1)	5.0(1)
4	-2.72(1)	5.848(1)	6.8(1)
5	0.95(2)	6.961(1)	6.4(1)

**Table 3:** Estimates of  $\Delta_4$  and  $\Delta'$  and the OPE coefficient  $\mathbf{p}_\phi$  from the truncated sum rule (3.4). These are, to the best of our knowledge, the only known estimates for these quantities.



**Figure 4:** Dispersion of the estimates of  $\Delta_4$  versus  $\Delta_\phi$  in the 2d Yang-Lee model. Notice the strong correlation between these two quantities.

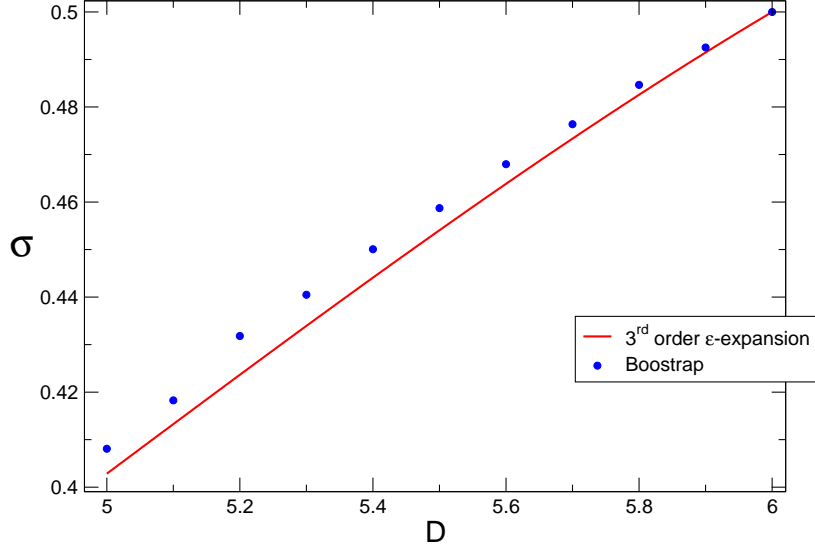
#### 4. 3d critical Ising model

We investigate the 3d critical Ising model with the same strategy applied to the Yang-Lee model, namely we start from the fusion rules of the free-field theory in the upper critical dimension –  $D = 4$  for a  $\phi^4$  theory – and see how this is modified in the non-trivial fixed point in  $4 - \epsilon$  dimensions. This time the first redundant operator is  $\phi^3$  which does not contribute to the OPE of  $\phi(x)\phi(y)$ , being an odd operator, so no further insight to simplify the fusion rule can be obtained (3.3). If we include the spin 4 operator, we cannot neglect the leading irrelevant scalar primary  $\varepsilon'$ , corresponding to  $\phi^4$ , since  $\Delta_{\phi^4} < \Delta_4$  in  $4 - \epsilon$  dimensions.

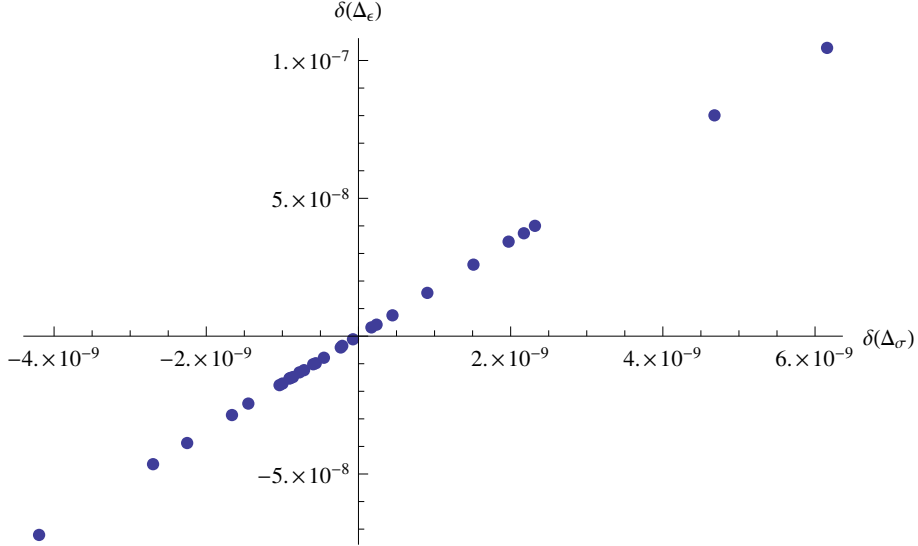
A first solution of the bootstrap constraints (1.2) is associated with the truncated fusion rule

$$[\Delta_\sigma] \times [\Delta_\sigma] = 1 + [\Delta_\varepsilon] + [\Delta_{\varepsilon'}] + [3, 2] + [\Delta_4, 4], \quad (4.1)$$

with  $\Delta_\sigma \equiv \Delta_\phi$ . It turns out that the bootstrap constraints (1.2) applied to the  $M = 5$  homogeneous equations (2.9) with  $m + n \leq 2$  are fulfilled. This time we do not have an isolated solution like in Yang-Lee model, but a one-parameter family of solutions. Using as input  $\Delta_4 = 5.0208(12)$ , which is the most precisely known scaling dimension of the 3d critical Ising model, we obtain a first rough estimate of the scaling dimensions of the operators involved in (4.1), namely  $\Delta_\sigma = 0.5145(3)$ ,  $\Delta_\varepsilon = 1.3735(15)$ ,  $\Delta_{\varepsilon'} = 3.80(1)$ .

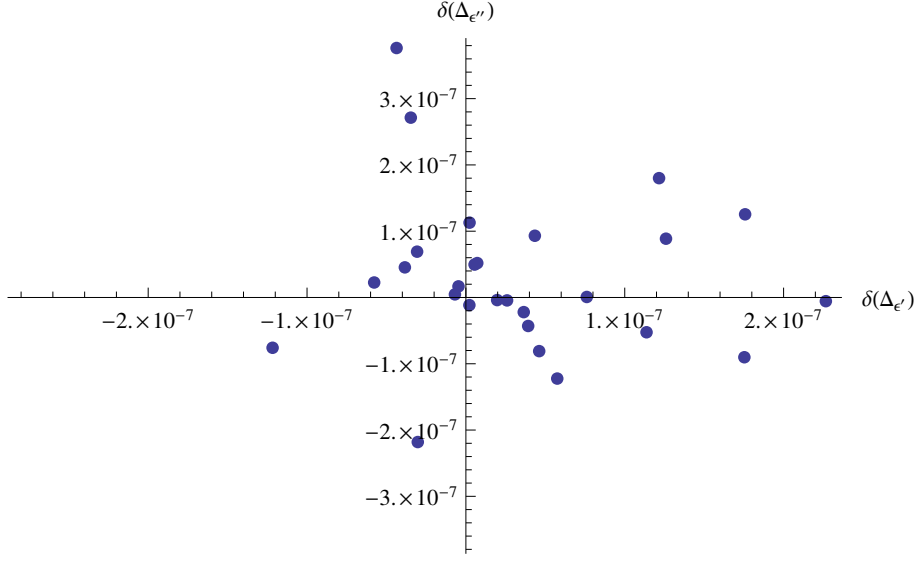


**Figure 5:** Plot of the function  $\sigma(D)$  given by eq. (3.5) compared with the results of our bootstrap analysis for  $5 \leq D \leq 6$ . In this range the different resummations of the epsilon expansion do not give visually different results. The difference between the bootstrap estimates and the epsilon expansion are probably due to the contributions of higher order in  $\epsilon$ .

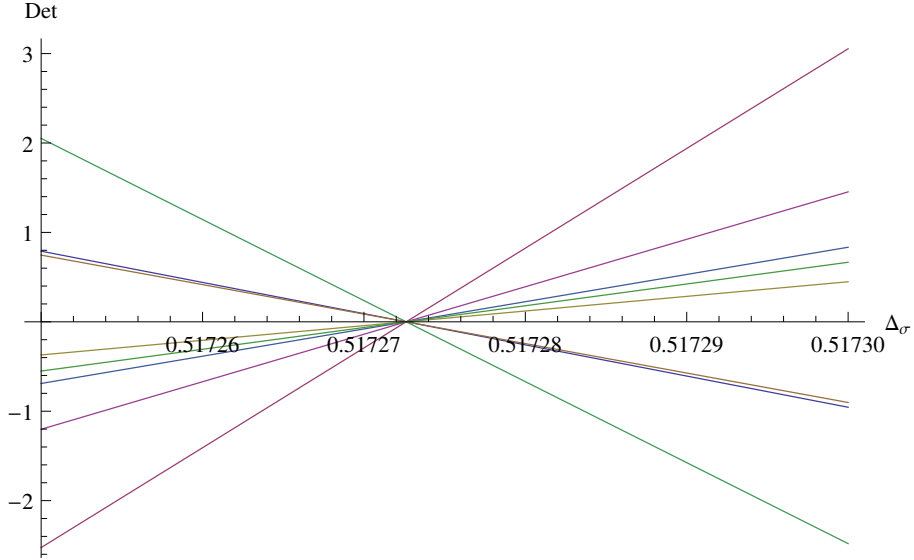


**Figure 6:** Dispersion of the solutions in the plane  $(\Delta_\sigma, \Delta_\epsilon)$  at  $\Delta_4 = 5.022$ . The origin coincides with the mean value of these quantities. Notice the microscopic value of the spread.

A better approximation is obtained including also a spin 6 operator, and, since in the  $\epsilon$  expansion the scaling dimensions of the fields  $\phi^6$  and  $\phi^8$  are smaller than  $\Delta_6$ , also these



**Figure 7:** Dispersion of the solutions in the plane  $(\Delta_{\varepsilon'}, \Delta_{\varepsilon''})$  at  $\Delta_4 = 5.022$ . The origin coincides with the mean value of these quantities.



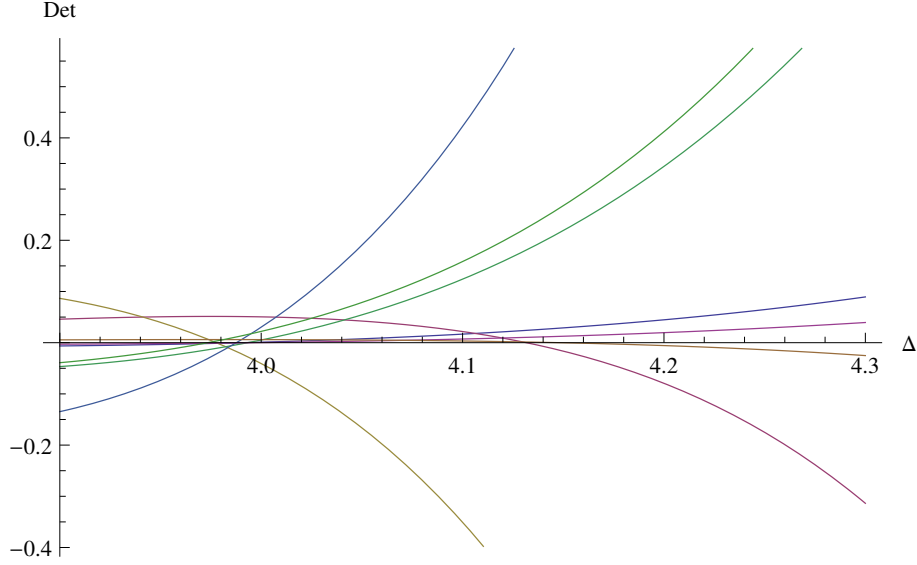
**Figure 8:** The zeros of the eight determinants associated to the solution (4.2) at  $\Delta = \Delta_\sigma$ , keeping all other scaling dimensions fixed at the values of the exact solution. In view of the microscopic spread of the solutions shown in figure 6, these different solutions are visually indistinguishable.

need to be included, resulting in the next to leading scalars  $\varepsilon'', \varepsilon'''$ .

Precisely we found that the truncated fusion rule

$$[\Delta_\sigma] \times [\Delta_\sigma] = 1 + [\Delta_\varepsilon] + [\Delta_{\varepsilon'}] + [\Delta_{\varepsilon''}] + [\Delta_{\varepsilon'''}] + [3, 2] + [\Delta_4, 4] + [\Delta_6, 6], \quad (4.2)$$

admits an exact solution of the bootstrap constraints (1.2) applied to the set of  $M = 8$  homogeneous equations (2.9) with  $m + n \leq 3$  and  $n < 3$ . The unknowns are the six quantities  $\Delta_\sigma, \Delta_\varepsilon, \Delta_{\varepsilon'}, \Delta_{\varepsilon''}, \Delta_{\varepsilon'''}, \Delta_6$ , so we can split the set of homogeneous equations in



**Figure 9:** Plot of the same determinants of figure 8 in the region  $\Delta \simeq 4$  where there is an approximate solution for  $\Delta_{\sigma'}$ . The spread of the zeros gives a rough estimate of the error.

28 consistent subsystems and as many bootstrap constraints (1.2) given by the vanishing of determinants of  $7 \times 7$  matrices. The corresponding spread of the solutions is far smaller (see figures 6 and 7) than the error of the input parameter  $\Delta_4$ .

There is a number of consistency checks that can be exploited to enlarge the spectrum of estimated scaling dimensions. We expect in particular that the leading irrelevant odd scalar  $\sigma'$  fulfils the same fusion rule as  $\sigma$ , at least at this truncation level<sup>2</sup>. This implies that if we treat  $\Delta_{\sigma}$  as a free parameter  $\Delta$ , keeping all the other scaling dimensions fixed, the 8 determinants which vanish at  $\Delta = \Delta_{\sigma}$  should also be approximately zero at  $\Delta = \Delta_{\sigma'}$ , and this actually happens in figures 8 and 9.

Similarly, we expected that the fusion rule of the energy operator  $\varepsilon(x)$  should coincide at this truncation level with that of  $\sigma(x)$ . However we did not find a corresponding solution. We found instead an approximate solution for the following enlarged fusion rule

$$[\Delta_{\varepsilon}] \times [\Delta_{\varepsilon}] = 1 + [\Delta_{\varepsilon}] + [\Delta_{\varepsilon'}] + [\Delta_{\varepsilon''}] + [\Delta_{\varepsilon'''}] + [3, 2] + [\Delta_2, 2] + [\Delta'_2, 2] + [\Delta_4, 4] + [\Delta'_4, 4] + [\Delta_6, 6]. \quad (4.3)$$

Here, besides all the operators of (4.2), contributions appear from three new spinning operators, namely two recurrences of spin 2 and one recurrence of spin 4. In this case the conformal bootstrap constraints are associated with determinants of  $10 \times 10$  matrices. Unlike the solution of (4.2), where the accuracy of the zeros was of the order  $10^{-16}$ , here we found only approximate zeros of the order  $10^{-2} - 10^{-3}$ . It is important to notice that the estimate of scaling dimensions of the operators appearing in both fusion rules almost coincide. The resulting set of estimates for the whole set of primary operators considered in our analysis is reported in tables 4 and 5. Finally, inserting these values in

<sup>2</sup>Since  $\phi^3$  is a redundant operator, this primary operator is associated with  $\phi^5$  in the renormalisation group analysis in  $4 - \epsilon$  dimensions.

the linear systems (2.8) and (1.1) we can extract the first OPE coefficients. In particular, inserting the values of our exact solution of (4.2) we find  $p_\varepsilon = \lambda_{\sigma\sigma\varepsilon}^2 = 0.62(1)$ . The central charge  $c$  turns out to be  $c/c_{free} = 0.98(1)$  which is slightly higher than the value  $c/c_{free} = 0.946600 + 0.000022 - 0.000015$  obtained with the assumption that this model saturates the upper unitarity bound [21].

scalar operators	$\sigma$	$\sigma'$	$\varepsilon$	$\varepsilon'$	$\varepsilon''$	$\varepsilon'''$
$\Delta$ , best estimates	0.51813(5)	$\gtrsim 4.5$	1.41275(25)	3.84(4)	4.67(11)	–
$\Delta$ , bootstrap	0.5171(1)	4.05(5)	1.413(2)	3.79(3)	4.60(7)	5.80(4)

**Table 4:** Low-lying scalar primary operators and their scaling dimensions obtained by the solution of the conformal bootstrap constraints (1.2) and compared with the best estimates, taken from [9] and [37].

spinning operators	$[\Delta_2, 2]$	$[\Delta'_2, 2]$	$[\Delta_4, 4]$	$[\Delta'_4, 4]$	$[\Delta_6, 6]$
$\Delta$ , best estimates	–	–	5.0208(12)	–	7.028(8)
$\Delta$ , bootstrap	5.117(1)	6.20(1)	input	6.70(1)	7.04(2)

**Table 5:** Low-lying primary operators with spin and their scaling dimensions obtained by the solution of the conformal bootstrap constraints (1.2). The input value for  $\Delta_4$  is taken from [37]. The estimate for the spin 6 operator is extracted from [19].

In conclusion, in this paper we have developed a general method implementing the crossing symmetry constraints in a large class of CFT, both unitary and not. Starting with the knowledge of the fusion rules, this method can generate systematically the correlation functions, by searching for the zeros of certain  $N \times N$  determinants made up of multiple derivatives of generalized hypergeometric functions. Such a study can be performed with the use of modest computing power, indeed all the results here presented have been obtained by using a single workstation. The method's application to the Yang -Lee edge singularity for  $2 \leq D \leq 6$  and to the 3d critical Ising model gives rather accurate results and can be further improved by enlarging the number  $N$  of primary operators included in the analysis.

#### *Added Note*

After this work was completed, we became aware of the preprint [38] where the low-lying spectrum of the 3d critical Ising model is precisely calculated under the assumption that this model saturates the upper unitarity bound. Assuming it lives at the change of slope of the boundary curve in  $(\Delta_\sigma, \Delta_\varepsilon)$  plane, very accurate values of these scaling dimensions are obtained, which are more precise than our estimates with the determinant method. On the contrary our method seems to give more predictive and stable results for primaries of higher scaling dimensions, as shown in Tables 4 and 5.

In the quoted paper it is reported strong numerical evidence that certain operators disappear from the spectrum as one approaches the putative 3d Ising point. Our method provides a simple explanation of this phenomenon. Indeed we directly study the solutions of the linear system (2.8)–(2.9), encoding the crossing symmetry constraint of a CFT,



whether unitary or not. By the knowledge of the operator spectrum of the theory it is possible to calculate the couplings  $\mathbf{p}_{\Delta_k}$  by simply solving the associated linear system. Assuming  $N$  equations and  $N$  unknowns, according to the Cramer's rule we have

$$\mathbf{p}_{\Delta_k} = \frac{\det f(\Delta_1, \Delta_2, \dots, \Delta_{k-1}, \Delta_{k+1}, \dots, \Delta_N)}{\det g(\Delta_1, \Delta_2, \dots, \Delta_k, \dots, \Delta_N)},$$

where the numerator is a  $(N-1) \times (N-1)$  minor described in eq. (1.2) and the denominator is the determinant of the linear system under study. It is important to notice that the numerator *does not depend on*  $\Delta_k$ . The unitarity bound arises because some of these couplings change sign. In other words, at the boundary of the parameter space allowed by unitarity some of these  $\mathbf{p}_{\Delta_k}$ 's vanish. These zeros, i.e. the vanishing of the minors (1.2), are just the constraints we used to evaluate the low-lying spectrum of the 3d critical Ising model.

## References

- [1] A. M. Polyakov, "Conformal symmetry of critical fluctuations", *JETP Lett.* **12** (1970) 381–383.
- [2] K. G. Wilson and J. B. Kogut, "The Renormalization group and the epsilon expansion," *Phys. Rept.* **12** (1974) 75.
- [3] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2**, 231 (1998) [hep-th/9711200].
- [4] S. Ferrara, A. F. Grillo, and R. Gatto, "Tensor representations of conformal algebra and conformally covariant operator product expansion", *Annals Phys.* **76** (1973) 161–188.
- [5] A. M. Polyakov, "Nonhamiltonian approach to conformal quantum field theory", *Zh. Eksp. Teor. Fiz.* **66** (1974) 23–42.
- [6] A. M. Polyakov, A. A. Belavin and A. B. Zamolodchikov, "Infinite Conformal Symmetry of Critical Fluctuations in Two-Dimensions," *J. Statist. Phys.* **34**, 763 (1984).
- [7] Y. Deng and H. W. J. Blöte, "Simultaneous analysis of several models in the three-dimensional Ising universality class," *Phys. Rev. E* **68** (2003) 036125.
- [8] M. Campostrini, A. Pelissetto, P. Rossi and E. Vicari, "25th-order high-temperature expansion results for three-dimensional Ising-like systems on the simple cubic lattice," *Phys. Rev. E* **65** (2002) 066127, [arXiv:cond-mat/0201180].
- [9] M. Hasenbusch, "A Finite Size Scaling Study of Lattice Models in the 3D Ising Universality Class" *Phys. Rev. B* **82** (2010) 174433. [arXiv:1004.4486]
- [10] C. Vafa, "Toward Classification Of Conformal Theories," *Phys. Lett. B* **206**, 421 (1988).
- [11] R. Rattazzi, V. S. Rychkov, E. Tonni and A. Vichi, "Bounding scalar operator dimensions in 4D CFT," *JHEP* **0812** (2008) 031 [arXiv:0807.0004 [hep-th]].
- [12] V. S. Rychkov and A. Vichi, "Universal Constraints on Conformal Operator Dimensions," *Phys. Rev. D* **80** (2009) 045006 [arXiv:0905.2211 [hep-th]].
- [13] R. Rattazzi, S. Rychkov and A. Vichi, "Central Charge Bounds in 4D Conformal Field Theory," *Phys. Rev. D* **83** (2011) 046011 [arXiv:1009.2725 [hep-th]].

- [14] D. Poland and D. Simmons-Duffin, “Bounds on 4D Conformal and Superconformal Field Theories,” *JHEP* **1105** (2011) 017 [arXiv:1009.2087 [hep-th]].
- [15] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, “Solving the 3D Ising Model with the Conformal Bootstrap,” *Phys. Rev. D* **86** (2012) 025022 [arXiv:1203.6064 [hep-th]].
- [16] D. Pappadopulo, S. Rychkov, J. Espin and R. Rattazzi, “OPE Convergence in Conformal Field Theory,” *Phys. Rev. D* **86** (2012) 105043 [arXiv:1208.6449 [hep-th]].
- [17] P. Liendo, L. Rastelli and B. C. van Rees, “The Bootstrap Program for Boundary  $CFT_d$ ,” arXiv:1210.4258 [hep-th].
- [18] S. El-Showk and M. F. Paulos, “Bootstrapping Conformal Field Theories with the Extremal Functional Method,” arXiv:1211.2810 [hep-th].
- [19] Z. Komargodski and A. Zhiboedov, “Convexity and Liberation at Large Spin,” *JHEP* **1311** (2013) 140 [arXiv:1212.4103 [hep-th]].
- [20] F. Gliozzi, “More constraining conformal bootstrap,” *Phys. Rev. Lett.* **111** (2013) 161602 [arXiv:1307.3111].
- [21] F. Kos, D. Poland and D. Simmons-Duffin, “Bootstrapping the  $O(N)$  Vector Models,” arXiv:1307.6856.
- [22] S. El-Showk, M. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, “Conformal Field Theories in Fractional Dimensions,” arXiv:1309.5089 [hep-th].
- [23] D. Gaiotto, D. Mazac and M. F. Paulos, “Bootstrapping the 3d Ising twist defect,” arXiv:1310.5078 [hep-th].
- [24] J. Polchinski, “Scale and Conformal Invariance in Quantum Field Theory,” *Nucl. Phys. B* **303** (1988) 226.
- [25] F. A. Dolan and H. Osborn, “Conformal partial waves and the operator product expansion,” *Nucl. Phys. B* **678** (2004) 491 [hep-th/0309180].
- [26] M. Billò, M. Caselle, D. Gaiotto, F. Gliozzi, M. Meineri and R. Pellegrini, “Line defects in the 3d Ising model,” *JHEP* **1307** (2013) 055 [arXiv:1304.4110 [hep-th]].
- [27] C. -N. Yang and T. D. Lee, “Statistical theory of equations of state and phase transitions. 1. Theory of condensation,” *Phys. Rev.* **87** (1952) 404.
- [28] T. D. Lee and C. -N. Yang, “Statistical theory of equations of state and phase transitions. 2. Lattice gas and Ising model,” *Phys. Rev.* **87** (1952) 410.
- [29] M. E. Fisher, “Yang-Lee Edge Singularity and  $\phi^3$  Field Theory,” *Phys. Rev. Lett.* **40** (1978) 1610.
- [30] G. Parisi and N. Sourlas, “Critical Behavior of Branched Polymers and the Lee-Yang Edge Singularity,” *Phys. Rev. Lett.* **46** (1981) 871.
- [31] Y. Park and M.E. Fisher, “Identity of the universal repulsive core singularity with Yang-Lee edge criticality,” *Phys. Rev. E* **60** (1999) 6323 Arxiv:con-mat/9907429
- [32] S.N.Lai and M. E. Fisher, *J. Chem. Phys.* **103** (1995) 8144.
- [33] H.P. Hsu, W. Nadler and P. Grassberger, “Simulation on lattice animals and trees”, *J. Phys. A* **38** (2005) 775

- [34] P. Butera and M. Pernici, “Yang-Lee edge singularities from extended activity expansions of the dimer density for bipartite lattices of dimensionality  $2 \leq d \leq 7$ ,” Phys. Rev. E **86**, 011104 (2012) [arXiv:1206.0872 [cond-mat.stat-mech]].
- [35] J. L. Cardy, “Conformal Invariance And The Yang-lee Edge Singularity In Two-dimensions,” Phys. Rev. Lett. **54**, 1354 (1985).
- [36] O. F. de Alcantara Bonfim, J.E. Kirkham and A.J. McKane, “Critical exponents to order  $\epsilon^3$  for  $\phi^3$  models of critical phenomena in  $6 - \epsilon$  dimensions”, J.Phys. A **13** (1980) L247
- [37] A. Pelissetto and E. Vicari, “Critical phenomena and Renormalization group theory,” Phys. Rept. **368** (2002) 549 [cond-mat/0012164].
- [38] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, “Solving the 3d Ising Model with the Conformal Bootstrap II. c-Minimization and Precise Critical Exponents,” arXiv:1403.4545 [hep-th].